A MODULAR BRANCHING RULE FOR THE GENERALIZED SYMMETRIC GROUPS

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Abstract. We give a modular branching rule for certain wreath products as a generalization of Kleshchev's modular branching rule for the symmetric groups. Our result contains a modular branching rule for the complex reflection groups $G(m,1,n)$ (which are often called the generalized symmetric groups) in splitting fields for $\mathbb{Z}/m\mathbb{Z}$. Especially for $m=2$ (which is the case of the Weyl groups of type B), we can give a modular branching rule in any field. Our proof is elementary in that it is essentially a combination of Frobenius reciprocity, Mackey theorem, Clifford's theory and Kleshchev's modular branching rule.

1. Introduction

Given a sequence of groups such as $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$, branching rule for this sequence is the rule that "describes" $\mathsf{Res}^{G_{n+1}}_{G_n}(V)$ for G_{n+1}-module V or $\mathsf{Ind}^{G_{n+1}}_{G_n}(W)$ for G_n-module W. Let us review the case of the symmetric groups $\{G_n = \mathfrak{S}_n\}_{n\geq 0}$ with the default embedding of $\mathfrak{S}_m$ into $\mathfrak{S}_n$ for $m < n$, that is the embedding with respect to the first m letters. In characteristic zero, we can summarize the classically known branching rule as follows (for the details, see Theorem 3.3).

- for any irreducible $\mathfrak{S}_{n+1}$-module V, $\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V)$ is multiplicity-free.
- Young's lattice controls the structure of $\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V)$ as $\mathfrak{S}_n$-module.

Here the meaning of the word "multiplicity-free" is not ambiguous because $\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V)$ is completely reducible in characteristic zero.

Recently, A.Kleshchev successfully discovered and proved its analogue in positive characteristics which is now known as Kleshchev's modular branching rule [Kl1, Kl2, Kl3, Kl4]. The language of quantum groups and Kashiwara's crystal bases [HK, Kas] lets us state Kleshchev's modular branching rule in chacateristic $p > 0$ succinctly and beautifully as follows (for the details, see Theorem 3.6).

- for any irreducible $\mathfrak{S}_{n+1}$-module V, $\mathsf{Soc}(\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V))$ is multiplicity-free.
- the crystal basis $B(\Lambda_0)$ of the fundamental irreducible $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$-module $L(\Lambda_0)$ controls the structure of $\mathsf{Soc}(\mathsf{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V))$ as $\mathfrak{S}_n$-module.

Here for an A-module X we denote by $\mathsf{Soc}(X)$ the largest completely reducible A-submodule of X.

We generalize the above Kleshchev's modular branching rule for the symmetric groups to certain wreath products. Let G be a finite group and F be its splitting field and further assume that any irreducible FG-module is 1-dimensional. We denote by α the number of inequivalent irreducible representations of G. For example,

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if G is a p-group then $\alpha = 1$. The main result of this paper is succinctly stated as follows (for the details, see Theorem 5.2).

- for any irreducible $G \wr \mathfrak{S}_{n+1}$-module V, $\mathsf{Soc}(\mathsf{Res}^{G \wr \mathfrak{S}_{n+1}}_{G \wr \mathfrak{S}_n}(V))$ is multiplicity-free.
- the crystal basis of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))^{\otimes \alpha}(\cong U_q(\mathfrak{g}((A^{(1)}_{p-1})^{\oplus \alpha})))$-module $L(\Lambda_0)^{\otimes \alpha}$ controls the structure of $\mathsf{Soc}(\mathsf{Res}^{G \wr \mathfrak{S}_{n+1}}_{G \wr \mathfrak{S}_n}(V))$ as $G \wr \mathfrak{S}_n$-module.

Here for simplicity we assume $\operatorname{char} F = p > 0$, but our result also contains the case of $\operatorname{char} F = 0$. Note that our result contains a modular branching rule for the complex reflection groups $G(m, 1, n) = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$ (which are often called the generalized symmetric groups) in splitting fields for $\mathbb{Z}/m\mathbb{Z}$. Especially for $m = 2$ (which is the case of the Weyl groups of type B), we can give a modular branching rule in any field.

We recall a known generalization of Kleshchev's modular branching rule. Let λ be a positive integral weight of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$ and consider the inductive system of the cyclotomic degenerate Hecke algebras $\{\mathcal{H}^{\lambda}_n\}_{n \geq 0}$ [Kl6, Chapter 7]. Then the following holds [Kl6]. See also [Ari, Bru, GV]. If we take $\lambda = \Lambda_0$ then we get Kleshchev's modular branching rule for the symmetric groups.

- for any irreducible $\mathcal{H}^{\lambda}_{n+1}$-module V, $\mathsf{Soc}(\mathsf{Res}^{\mathcal{H}^{\lambda}_{n+1}}_{\mathcal{H}^{\lambda}_n}(V))$ is multiplicity-free.
- the crystal basis $B(\lambda)$ of $U_q(\mathfrak{g}(A^{(1)}_{p-1}))$-module $L(\lambda)$ controls the structure of $\mathsf{Soc}(\mathsf{Res}^{\mathcal{H}^{\lambda}_{n+1}}_{\mathcal{H}^{\lambda}_n}(V))$ as $\mathcal{H}^{\lambda}_n$-module.

It should be noted that the different crystals appear in modular branching for the wreath products and for the Hecke algebras. Although known proofs for the Hecke algebra case require the recent advances in modular representation theory actively involving other areas such as quantum groups, our case is elementary in that our proof is essentially a combination of Frobenius reciprocity, Mackey theorem, Clifford's theory and Kleshchev's modular branching rule.

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Notations and conventions In the following discussion of this paper, we assume for simplicity that any group is finite and any module is finite dimensional.

- For a finite dimensional algebra A, we denote by $\mathsf{Irr}(A)$ the set of isomorphism classes of irreducible A-modules.
- For an A-module V, we denote by $[V]_A$ the isomorphism class of A-modules isomorphic to V. If A is clear from the context, we often omit the suffix.
- When we say directed graphs without modifiers, it means directed graphs with no loops nor no multiple-egdes. We write directed graph as $X = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is its adjacent relation meaning there is a directed arrow from v_1 to v_2 if and only if $(v_1, v_2) \in E$.
- Let $l \geq 2$ be a positive integer. A partition $(\lambda_1, \cdots, \lambda_k)$ is called l-restricted if $\lambda_i - \lambda_{i+1} < l$ for any $1 \leq i < k$. It is called l-regular if its conjugate partition is l-restricted. All the partitions are defined to be 0-restricted and 0-regular.

- Let $p > 0$ be a prime number and G be a group. A conjugacy class of G is called p-regular if the order of (one of or equivalently all of) its elements is prime to p. All the congugacy classes of G are defined to be 0-regular.
- For a given sequence of non-negative integers $\overrightarrow{\boldsymbol{n}} = (n_1, \cdots, n_\alpha)$ such that $\sum_{\beta=1}^{\alpha} n_\beta = n$ (such $\overrightarrow{\boldsymbol{n}}$ is called a composition of n), we denote by $\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}$ the Young subgroup of $\mathfrak{S}_n$, that is

$$\begin{aligned}\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}} &\stackrel{\text{def}}{=} \mathfrak{S}_{\{1,\cdots,n_1\}} \times \mathfrak{S}_{\{n_1+1,\cdots,n_1+n_2\}} \times \cdots \times \mathfrak{S}_{\{n-n_\beta+1,\cdots,n\}} \\ &\cong \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_\beta}.\end{aligned}$$

2. Definitions

Definition 2.1. *Given a field F and an inductive system of groups $\mathfrak{I} = (\{G_n\}_{n\geq 0}, \{\varphi_n : G_n \to G_{n+1}\}_{n\geq 0})$, we define **modular branching graph** $\mathbb{B}_F(\mathfrak{I})$, which is a priori a directed graph with multiple-edges, as follows.*

- *the vertices are the elements of* $\bigsqcup_{n\geq 0} \mathsf{Irr}(FG_n)$.
- *for two vertices $[W] \in \mathsf{Irr}(FG_n)$ and $[V] \in \mathsf{Irr}(FG_{n+1})$ of $\mathbb{B}_F(\mathfrak{I})$, there are* $\dim \mathrm{Hom}_{FG_n}(W, \mathsf{Res}_{FG_n}^{FG_{n+1}} V)$ *edges from $[W]$ to $[V]$.*

*If $\mathbb{B}_F(\mathfrak{I})$ has only single-edges, we say that $\mathfrak{I}$ is **socle multiplicity-free** over F.*

Remark 2.2. *Note that for FG_n-module W and FG_{n+1}-module V we have*

$$\begin{aligned}&\dim \mathrm{Hom}_{G_{n+1}}(\mathsf{Ind}_{G_n}^{G_{n+1}}(W), V) = \dim \mathrm{Hom}_{G_n}(W, \mathsf{Res}_{G_n}^{G_{n+1}}(V)) \\ &= \dim \mathrm{Hom}_{G_{n+1}}(V^*, \mathsf{Ind}_{G_n}^{G_{n+1}}(W^*)) = \dim \mathrm{Hom}_{G_n}(\mathsf{Res}_{G_n}^{G_{n+1}}(V^*), W^*)\end{aligned}$$

by Frobenius reciprocity and Nakayama relations. Hence the natural 4 choices for the definition of modular branching graph all coincide with each other if all the irreducible FG_n-modules are self-dual for each n.

Remark 2.3. *There is yet another natural definition of modular branching graph that replaces* $\dim \mathrm{Hom}_{G_n}(W, \mathsf{Res}_{G_n}^{G_{n+1}} V)$ *by* $[\mathsf{Res}_{G_n}^{G_{n+1}} V : W]$. *For the symmetric groups case $\{G_n = \mathfrak{S}_n\}_{n\geq 0}$, Kleshchev partially succeeded in describing this* [Kl7] *(and as a corollary we know that the decomposition number of $\mathfrak{S}_n$ in positive characteristics can be arbitrary large), and Kleshchev also showed that knowing this type of modular branching graph is as hard as knowing the decomposition numbers* [Kl5].

Definition 2.4. *Let $X_1 = (V_1, E_1), \cdots, X_k = (V_k, E_k)$ be k directed graphs. We define the directed graph $X_1 * \cdots * X_k = (V, E)$ as follows.*

- $V = V_1 \times \cdots \times V_k$.
- $((v_1, \cdots, v_k), (w_1, \cdots, w_k)) \in E$ *iff there exists unique $1 \leq j \leq k$ such that $(v_j, w_j) \in E_j$ and $v_{j'} = w_{j'}$ for all $j' \neq j$.*

Especially when $X_1 = \cdots = X_k = X$, we write $X_1 * \cdots * X_k = X^{*k}$.

3. Representation theory of the symmetric groups

We shall first recall the representation theory of the symmetric groups. It is well-known that for each partition λ of n, we can construct $\mathbb{Z}$-free, $\mathbb{Z}$-finite rank, $\mathbb{Z}\mathfrak{S}_n$-module S^λ which is called the **Specht module** [Jam, Chapter 4]. Each S^λ has an $\mathfrak{S}_n$-invariant symmetric bilinear form. For any field F, we write $S_F^\lambda = F \otimes_{\mathbb{Z}} S^\lambda$ and denote by D_F^λ the quotient of S_F^λ by the radical of its invariant form. It is also

well-known that $D_F^\lambda = S_F^\lambda$ if $\operatorname{char} F = 0$ [Jam, Chapter 4]. The following is the fundamental theorem of the representation theory of the symmetric groups.

Theorem 3.1 ([Jam, Theorem 11.5])**.** *Suppose that our ground field F has characteristic $p(\geq 0)$. As μ varies over p-regular partitions of n, D_F^μ varies over a complete set of inequivalent irreducible $F\mathfrak{S}_n$-modules. Each D_F^μ is self-dual and absolutely irreducible. Every field is splitting field for $\mathfrak{S}_n$.*

Definition 3.2. *We denote by $\mathfrak{S}$ the inductive system of the symmetric groups*

$$\mathfrak{S} = (\mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots)$$

Here if $m < n$, the default embedding of $\mathfrak{S}_m$ into $\mathfrak{S}_n$ is with respect to the first m letters.

Theorem 3.3 ([Jam, Theorem 9.2])**.** *For any field F of characteristic zero, $\mathfrak{S}$ is socle multiplicity-free over F. Moreover, the map $\mathbb{B}_0 \to \mathbb{B}_F(\mathfrak{S}), \lambda \mapsto [S^\lambda]$ is an isomorphism as directed graphs where $\mathbb{B}_0 = (\mathcal{P}_0, E_0)$ is the directed graph obtained from the Young's lattice $\mathcal{P} \stackrel{def}{=} \{\lambda \vdash n \mid n \geq 0\} (= \mathcal{P}_0)$ in the trivial manner.*

Theorem 3.3 is known as the classical branching rule. Its analog in positive characteristics is known as Kleshchev's modular branching rule. First we introduce the necessary terminology.

Definition 3.4. *Let $l \geq 2$ be a positive integer and $i \in \mathbb{Z}/l\mathbb{Z}$. Note that the following definitions depend on this given l. Let λ be a partition. We identify the partition λ with the Young diagram of shape λ, that is $\{(r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} | s \leq \lambda_r\}$.*

(1) *For a node $A = (r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, we define its **residue** to be $-r+s+l\mathbb{Z} \in \mathbb{Z}/l\mathbb{Z}$.*

(2) *A node A inside λ is called i-**removable** if the residue of A equals i and $\lambda \setminus \{A\}$ is still a Young diagram.*

(3) *A node A outside λ is called i-**addable** if the residue of A equals i and $\lambda \cup \{A\}$ is again a Young diagram.*

(4) *Now label all i-addable nodes of λ by $+$ and all i-removable nodes of λ by $-$. The i-**signature** of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs.*

(5) *The **reduced** i-**signature** of λ is the sequence of pluses and minuses obtained from the i-signature of λ by successively erasing all neighboring pairs of the form $-+$. Note that the reduced i-signature of λ always looks like a sequence that starts with $+$s followed by $-$s.*

(6) *Nodes that correspond to $+$s of the reduced i-signature of λ are called i-**conormal**.*

(7) *The node that corresponds to the rightmost $+$ of the reduced i-signature of λ is called i-**cogood**. It is the rightmost i-conormal node.*

(8) *We set $\varphi_i(\lambda) = \#\{i\text{-conormal nodes of } \lambda\}$.*

(9) *If $\varphi_i(\lambda) > 0$, we set $\tilde{f}_i(\lambda) = \lambda \cup \{A\}$ where A is the (unique) i-cogood node.*

Definition 3.5. *Let p be a prime number. We define the directed graph $\mathbb{B}_p = (\mathcal{P}_p, E_p)$ where $\mathcal{P}_p = \{\lambda \in \mathcal{P} \mid \lambda \text{ is } p\text{-regular}\}$ and the adjacent relation $E_p = \{(\lambda, \mu) \in \mathcal{P}_p \times \mathcal{P}_p \mid \exists i \in \mathbb{Z}/p\mathbb{Z}, \varphi_i(\lambda) > 0 \text{ and } \tilde{f}_i(\lambda) = \mu\}$.*

The following is the form of Kleshchev's modular branching rule that we use.

Theorem 3.6 ([Kl2, Kl6])**.** *For any field F whose characteristic is $p > 0$, $\mathfrak{S}$ is socle multiplicity-free over F. Moreover, the map $\mathbb{B}_p \to \mathbb{B}_F(\mathfrak{S}), \lambda \mapsto [D_F^\lambda]$ is an isomorphism as directed graphs.*

4. Representation theory of wreath products

Let G be a group and F be a splitting field for G. We write $\mathsf{Irr}(FG) = \{[V_1], \cdots, [V_\alpha]\}$. Definitions in this chapter depend on these datum.

Recall that the wreath product $G \wr \mathfrak{S}_n$ is the semi-direct product $G^n \rtimes_\theta \mathfrak{S}_n$ where

$$\theta : \mathfrak{S}_n \longrightarrow \mathrm{Aut}(G^n), \quad \sigma \mapsto \theta(\sigma)((g_1, \cdots, g_n)) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)}).$$

Hence any element $x \in G \wr \mathfrak{S}_n$ is written $x = (f; \pi)$ for uniquely determined $f = (g_1, \cdots, g_n) \in G^n$ and $\sigma \in \mathfrak{S}_n$ and the multiplication rule is

$$(f_1; \sigma_1) \cdot (f_2; \sigma_2) = (f_1 \cdot (f_2)_{\sigma_1}; \sigma_1 \cdot \sigma_2) \tag{1}$$

where $(f_2)_{\sigma_1} \stackrel{\text{def}}{=} \theta(\sigma_1)(f_2)$. The normal subgroup $(G^n \cong)\{(f; 1_{\mathfrak{S}_n}) \mid f \in G^n\} \subseteq G \wr \mathfrak{S}_n$ is often called the **base group** of $G \wr \mathfrak{S}_n$ and denoted by G^*.

The representation theory of wreath products is a typical application of Clifford's theory and well-presented in [JK, Chapter 4]. This chapter is a very brief summary of [JK, Chapter 4]. Note that notations and the assumptions for definitions or theorems are slightly changed.

Definition 4.1. *For a given composition $\overrightarrow{\boldsymbol{n}} = (n_1, \cdots, n_\alpha)$ of n and a permutation $\pi \in \mathfrak{S}_\alpha$, we define irreducible FG^*-module $E(\overrightarrow{\boldsymbol{n}}; \pi) = V_{\pi(1)}^{\otimes n_1} \otimes \cdots \otimes V_{\pi(\alpha)}^{\otimes n_\alpha}$. We usually omit π when $\pi = 1_{\mathfrak{S}_\alpha}$.*

Theorem 4.2 ([JK, 4.3.27])**.** *Let $\overrightarrow{\boldsymbol{n}}$ be as above. The inertia group for $E(\overrightarrow{\boldsymbol{n}})$ is given by $\{(f; \sigma) \mid f \in G^n, \sigma \in \mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}\} (\subseteq G \wr \mathfrak{S}_n)$. We denote this group by $G^*\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}$.*

Definition 4.3 ([JK, p.154])**.** *Let $\overrightarrow{\boldsymbol{n}}$ and π be as above. To the underlying vector space $E(\overrightarrow{\boldsymbol{n}}; \pi)$, we can define a $F[G^*\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}]$-module structure by*

$$(f; \sigma)(v_1 \otimes \cdots \otimes v_n) \stackrel{def}{=} g_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma^{-1}}(n)$$

where $f = (g_1, \cdots, g_n) \in G^n$ and $\sigma \in \mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}$. We denote this module by $\widetilde{E}(\overrightarrow{\boldsymbol{n}}; \pi)$. We usually omit π when $\pi = 1_{\mathfrak{S}_\alpha}$.

Definition 4.4. *For a given sequence of p-regular partitions $\overrightarrow{\boldsymbol{\lambda}} = (\lambda_1, \cdots, \lambda_\alpha)$ such that $\sum_{\beta=1}^{\alpha} |\lambda_\beta| = n$. We write $\overrightarrow{\boldsymbol{n}} = (|\lambda_1|, \cdots, |\lambda_\alpha|)$ and define a irreducible $F\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}$-module $D(\overrightarrow{\boldsymbol{\lambda}}) = D_F^{\lambda_1} \otimes \cdots \otimes D_F^{\lambda_\alpha}$.*

Definition 4.5 ([JK, 4.3.31])**.** *Let $\overrightarrow{\boldsymbol{\lambda}}$ and $\overrightarrow{\boldsymbol{n}}$ be as above. To the underlying vector space $D(\overrightarrow{\boldsymbol{\lambda}})$, we define a $F[G^*\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}]$-module structure by*

$$(f; \sigma)(w_1 \otimes \cdots \otimes w_\alpha) \stackrel{def}{=} \sigma(w_1 \otimes \cdots \otimes w_\alpha)$$

where $f = (g_1, \cdots, g_n) \in G^n$ and $\sigma \in \mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}$. We denote this module by $\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})$.

Definition and Theorem 4.6 ([JK, 4.4.3])**.** *Let $\overrightarrow{\boldsymbol{\lambda}}$ and $\overrightarrow{\boldsymbol{n}}$ be as above. We define a $F[G \wr \mathfrak{S}_n]$-module $C(\overrightarrow{\boldsymbol{\lambda}})$ by*

$$C(\overrightarrow{\boldsymbol{\lambda}}) = \mathsf{Ind}_{F[G^*\mathfrak{S}_{\overrightarrow{\boldsymbol{n}}}]}^{F[G \wr \mathfrak{S}_n]} \widetilde{E}(\overrightarrow{\boldsymbol{n}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}).$$

If $\overrightarrow{\boldsymbol{\lambda}} = (\lambda_1, \cdots, \lambda_\alpha) \in \mathcal{P}_p^\alpha$ varies while satisfying $\sum_{\beta=1}^{\alpha} |\lambda_\beta| = n$, $C(\overrightarrow{\boldsymbol{\lambda}})$ varies over a complete set of inequivalent irreducible $F[G \wr \mathfrak{S}_n]$-modules.

5. Main Results

Definition 5.1. *Denote by $G \wr \mathfrak{S}$ the following inductive system of the wreath product groups of G.*

$$G \wr \mathfrak{S} = (G \wr \mathfrak{S}_0 \subseteq G \wr \mathfrak{S}_1 \subseteq G \wr \mathfrak{S}_2 \subseteq \cdots).$$

Theorem 5.2. *Let G be a group and F be its splitting filed of characteristic $p(\geq 0)$ such that any irreducible FG-module is 1-dimensional. We denote by α the number of p-regular conjugacy classes of G and fix a labeling of $\mathsf{Irr}(FG) = \{[V_1], \cdots, [V_\alpha]\}$. Then $G \wr \mathfrak{S}$ is socle multiplicity-free over F. Moreover, the map*

$$\mathbb{B}_p^{*\alpha} \longrightarrow \mathbb{B}_F(G \wr \mathfrak{S}), \quad (\lambda_1, \cdots, \lambda_\alpha) \longmapsto [C(\lambda_1, \cdots, \lambda_\alpha)]$$

is an isomorphism as directed graphs.

Proof. We show the equivalent statement that for any sequence of p-regular partitions $\overrightarrow{\boldsymbol{\lambda}} = (\lambda_1, \cdots, \lambda_\alpha)$ and $\overrightarrow{\boldsymbol{\mu}} = (\mu_1, \cdots, \mu_\alpha)$ such that $\sum_{i=1}^{\alpha} |\lambda_i| = n, \sum_{i=1}^{\alpha} |\mu_i| = n+1$, we have

$$\dim \mathrm{Hom}_{G\wr\mathfrak{S}_n}(C(\overrightarrow{\boldsymbol{\lambda}}), \mathsf{Res}^{G\wr\mathfrak{S}_{n+1}}_{G\wr\mathfrak{S}_n} C(\overrightarrow{\boldsymbol{\mu}})) \leq 1 \tag{2}$$

and the equality holds exactly when the following condition (3) holds.

$$\text{there exists unique } 1 \leq \gamma \leq \alpha \text{ s.t.} \begin{cases} (\lambda_\gamma, \mu_\gamma) \in E_p \text{ (recall } \mathbb{B}_p = (\mathcal{P}_p, E_p)). \\ \text{for any } \gamma' \neq \gamma \text{ we have } \lambda_{\gamma'} = \mu_{\gamma'}. \end{cases} \tag{3}$$

Let $\overrightarrow{\boldsymbol{a}} = (|\lambda_1|, \cdots, |\lambda_\alpha|)$, $\overrightarrow{\boldsymbol{b}} = (|\mu_1|, \cdots, |\mu_\alpha|)$, $\overrightarrow{\boldsymbol{c}} = (|\lambda_1|, \cdots, |\lambda_\alpha|, 1)$ and let $X = G \wr \mathfrak{S}_{n+1}, Y = G^{n+1}\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}(\subseteq X), Z = G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}(\subseteq X), W = G \wr \mathfrak{S}_n$. To avoid possible confusion, we reserve the trivial group isomorphism $t : (W \supseteq) G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}} \xrightarrow{\sim} Z$. By Frobenius reciprocity,

$$\begin{aligned}
\dim \mathrm{Hom}_W(C(\overrightarrow{\boldsymbol{\lambda}}), \mathsf{Res}^X_W C(\overrightarrow{\boldsymbol{\mu}})) &= \dim \mathrm{Hom}_W(\mathsf{Ind}^W_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}), \mathsf{Res}^X_W C(\overrightarrow{\boldsymbol{\mu}})) \\
&= \dim \mathrm{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}), \mathsf{Res}^W_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \mathsf{Res}^X_W C(\overrightarrow{\boldsymbol{\mu}})) \\
&= \dim \mathrm{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}), \mathsf{Res}^X_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} C(\overrightarrow{\boldsymbol{\mu}})).
\end{aligned}$$

Let D be a (Z, Y)-double coset representatives in X. By Mackey theorem,

$$\begin{aligned}
\mathsf{Res}^X_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} C(\overrightarrow{\boldsymbol{\mu}}) &= {}^t \mathsf{Res}^X_Z C(\overrightarrow{\boldsymbol{\mu}}) \\
&= {}^t \mathsf{Res}^X_Z \mathsf{Ind}^X_Y(\widetilde{E}(\overrightarrow{\boldsymbol{b}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})) \\
&\cong \bigoplus_{d \in D} {}^t \mathsf{Ind}^Z_{dYd^{-1}\cap Z} {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd} \widetilde{E}(\overrightarrow{\boldsymbol{b}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}}))
\end{aligned}$$

as $F[G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}]$-modules where ${}^dM$ for $F[Y\cap d^{-1}Zd]$-module M stands for a $F[dYd^{-1}\cap Z]$-module which is obtained by the pullback through the group isomorpshim

$$\varrho_d : dYd^{-1} \cap Z \xrightarrow{\sim} Y \cap d^{-1}Zd, \quad x \mapsto d^{-1}xd$$

and the same for t.

Now we recall a necessary fact about the double coset representatives of the symmetric groups. Let $\overrightarrow{\boldsymbol{\nu}_1}, \overrightarrow{\boldsymbol{\nu}_2}$ be compositions of n. Denote by $D_{\overrightarrow{\boldsymbol{\nu}_2}}$ the set of minimal length left $\mathfrak{S}_{\overrightarrow{\boldsymbol{\nu}_2}}$-coset representatives in $\mathfrak{S}_n$. Then $D_{\overrightarrow{\boldsymbol{\nu}_1}\overrightarrow{\boldsymbol{\nu}_2}} \stackrel{\text{def}}{=} D^{-1}_{\overrightarrow{\boldsymbol{\nu}_1}} \cap D_{\overrightarrow{\boldsymbol{\nu}_2}}$ is the set of minimal length $(\mathfrak{S}_{\overrightarrow{\boldsymbol{\nu}_1}}, \mathfrak{S}_{\overrightarrow{\boldsymbol{\nu}_2}})$-double coset representatives in $\mathfrak{S}_n$ [DJ]. In

the following discussion, it is important that we know $D_{\overrightarrow{\boldsymbol{\nu}_2}}$ explicitly, that is for $\overrightarrow{\boldsymbol{\nu}_2} = (\eta_1, \cdots, \eta_\kappa)$ we have

$$D_{\overrightarrow{\boldsymbol{\nu}_2}} = \left\{ \sigma \in \mathfrak{S}_n \,\middle|\, \begin{smallmatrix} \sigma(1)<\sigma(2)<\cdots<\sigma(\eta_1) \\ \vdots \\ \sigma(n-\eta_{\kappa-1}+1)<\cdots<\sigma(n) \end{smallmatrix} \right\}. \tag{4}$$

By the multiplication rule (1), it is clear that $D_{\overrightarrow{\boldsymbol{c}}\,\overrightarrow{\boldsymbol{b}}} (\subseteq \mathfrak{S}_{n+1} \subseteq X)$ is a (Z, Y)-double coset representatives in X.

So we need to compute for each $d \in D_{\overrightarrow{\boldsymbol{c}}\,\overrightarrow{\boldsymbol{b}}}$,

$$L(d) \overset{\text{def}}{=} \dim \operatorname{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \left(\widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}), {}^t\,\mathsf{Ind}^Z_{dYd^{-1}\cap Z}\, {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})) \right).$$

We compute this value by taking their (various types of) duals. Note that dual operation induces an involution $\mathsf{Irr}(FG) \to \mathsf{Irr}(FG)$. We denote it by $\tau \in \mathfrak{S}_\alpha$ meaning that $[V_i^*] = [V_{\tau(i)}]$ for any $1 \le i \le \alpha$.

Sublemma 5.3.
(P) $\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})^* \cong \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})$ *as* $F[G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}]$*-module.*
(Q) $\widetilde{E}(\overrightarrow{\boldsymbol{a}})^* \cong \widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau)$ *as* $F[G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}]$*-module and* $\widetilde{E}(\overrightarrow{\boldsymbol{b}})^* \cong \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau)$ *as* $F[G^{n+1}\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}]$*-module.*

Proof.
(P) By Theorem 3.1, there exists $\mathfrak{S}_{|\lambda_k|}$-module isomorpshim $\Psi_k : D_F^{\lambda_k} \xrightarrow{\sim} (D_F^{\lambda_k})^*$ for each $1 \le k \le \alpha$. Then it is easy to check that the composition

$$D(\overrightarrow{\boldsymbol{\lambda}}) \xrightarrow[\Psi_1\otimes\cdots\otimes\Psi_\alpha]{\sim} (D_F^{\lambda_1})^* \otimes \cdots \otimes (D_F^{\lambda_\alpha})^* \xrightarrow[\text{can}]{\sim} D(\overrightarrow{\boldsymbol{\lambda}})^*$$

induces an $F[G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}]$-module isomorphism $\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}) \xrightarrow{\sim} \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})^*$ (see Definition 4.5).
(Q) The same as in (P) (see Definition 4.3). $\square$

Therefore, for each $d \in D_{\overrightarrow{\boldsymbol{c}}\,\overrightarrow{\boldsymbol{b}}}$,

(5)

$$\begin{aligned} L(d) &= \dim \operatorname{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \left(\widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}), {}^t\,\mathsf{Ind}^Z_{dYd^{-1}\cap Z}\, {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})) \right) \\ &= \dim \operatorname{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \left(({}^t\,\mathsf{Ind}^Z_{dYd^{-1}\cap Z}\, {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})))^*, (\widetilde{E}(\overrightarrow{\boldsymbol{a}}) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}))^* \right) \\ &= \dim \operatorname{Hom}_{G^n\mathfrak{S}_{\overrightarrow{\boldsymbol{a}}}} \left({}^t\,\mathsf{Ind}^Z_{dYd^{-1}\cap Z}\, {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})), \widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}) \right) \\ &= \dim \operatorname{Hom}_{dYd^{-1}\cap Z} \left({}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})), \mathsf{Res}^Z_{dYd^{-1}\cap Z}\, {}^{t^{-1}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})) \right). \end{aligned}$$

Note that $dYd^{-1} \cap Z = G^n(d\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}d^{-1} \cap \mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}) (\subseteq X)$. By restricting to the subgroup $G^n \subseteq dYd^{-1} \cap Z$, we have

$$\begin{aligned} L(d) &\le \dim \operatorname{Hom}_{G^n} \left(\mathsf{Res}^{dYd^{-1}\cap Z}_{G^n}\, {}^d(\mathsf{Res}^Y_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})), \mathsf{Res}^Z_{G^n}\, {}^{t^{-1}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})) \right) \\ &= \dim \operatorname{Hom}_{G^n} \left((V_{\tau(\xi(d^{-1}(1)))} \otimes \cdots \otimes V_{\tau(\xi(d^{-1}(n)))})^{\oplus \dim \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})}, E(\overrightarrow{\boldsymbol{a}};\tau)^{\oplus \dim \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})} \right) \end{aligned}$$

where we denote by $\xi(\chi)$ for $1 \le \chi \le n+1$ the unique $1 \le \xi \le \alpha$ such that

$$|\mu_1| + \cdots + |\mu_{\xi-1}| < \chi \le |\mu_1| + \cdots + |\mu_\xi|.$$

Sublemma 5.4. *If* $L(d) > 0$*, then there exists unique* $1 \le j \le \alpha$ *such that the followings are met.*
(A) $|\lambda_{j'}| = |\mu_{j'}|$ *for any* $j' \ne j$*.*

(B) $|\lambda_j|+1=|\mu_j|$.
(C) $d(|\lambda_1|+\cdots+|\lambda_j|+1)=n+1$.
(D) $d^{-1}(1)<\cdots<d^{-1}(n)$.

Moreover, we have

$$
\begin{aligned}
U_{1,d} &\overset{def}{=} d\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}d^{-1}\cap\mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}=\mathfrak{S}_{(|\mu_1|,\cdots,|\mu_{j-1}|,|\mu_j|-1,|\mu_{j+1}|,\cdots,|\mu_\alpha|,1)},\\
U_{2,d} &\overset{def}{=} \mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}\cap d^{-1}\mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}d=\mathfrak{S}_{(|\mu_1|,\cdots,|\mu_{j-1}|,|\mu_j|-1,1,|\mu_{j+1}|,\cdots,|\mu_\alpha|)}.
\end{aligned}\tag{6}
$$

Proof. We show that $j=\xi(d^{-1}(n+1))$. Note that

$$
\begin{aligned}
&\dim\mathrm{Hom}_{G^n}\left((V_{\tau(\xi(d^{-1}(1)))}\otimes\cdots\otimes V_{\tau(\xi(d^{-1}(n)))})^{\oplus\dim\widetilde{D}(\overrightarrow{\boldsymbol{\mu}})},E(\overrightarrow{\boldsymbol{a}};\tau)^{\oplus\dim\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})}\right)\\
&=\dim\widetilde{D}(\overrightarrow{\boldsymbol{\mu}})\dim\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})\dim\mathrm{Hom}_{G^n}\left((V_{\tau(\xi(d^{-1}(1)))}\otimes\cdots\otimes V_{\tau(\xi(d^{-1}(n)))}),E(\overrightarrow{\boldsymbol{a}};\tau)\right)
\end{aligned}
$$

and the isomorphim between F-vector spaces

$$
\begin{aligned}
&\mathrm{Hom}_{G^n}\left((V_{\tau(\xi(d^{-1}(1)))}\otimes\cdots\otimes V_{\tau(\xi(d^{-1}(n)))}),E(\overrightarrow{\boldsymbol{a}};\tau)\right)\\
&\cong\mathrm{Hom}_G(V_{\tau(\xi(d^{-1}(1)))},V_{\tau(\zeta(1))})\otimes\cdots\otimes\mathrm{Hom}_G(V_{\tau(\xi(d^{-1}(n)))},V_{\tau(\zeta(n))}).
\end{aligned}\tag{7}
$$

where we denote by $\zeta(\chi)$ for $1\le\chi\le n$ the unique $1\le\zeta\le\alpha$ such that

$$|\lambda_1|+\cdots+|\lambda_{\zeta-1}|<\chi\le|\lambda_1|+\cdots+|\lambda_\zeta|.$$

Hence, if $L(d)>0$, we have (by recalling $d\in D_{\overrightarrow{\boldsymbol{c}}}^{-1}\cap D_{\overrightarrow{\boldsymbol{b}}}\subseteq D_{\overrightarrow{\boldsymbol{c}}}^{-1}$)

$$
\begin{cases}
1\le d^{-1}(1)<\cdots<d^{-1}(|\lambda_1|)\le|\mu_1|\\
|\mu_1|+1\le d^{-1}(|\lambda_1|+1)<\cdots<d^{-1}(|\lambda_1|+|\lambda_2|)\le|\mu_1|+|\mu_2|\\
\qquad\vdots\\
(n+1)-|\mu_\alpha|+1\le d^{-1}(n-|\lambda_\alpha|+1)<\cdots<d^{-1}(n)\le n+1.
\end{cases}
$$

This implies that we have (A) and (B). Hence it is enough to show that (C) holds. Suppose to the contrary, we have

$$|\mu_1|+\cdots+|\mu_{j-1}|+1\le d^{-1}(n+1)\le|\mu_1|+\cdots+|\mu_j|-1.$$

Because $d\in D_{\overrightarrow{\boldsymbol{c}}}^{-1}\cap D_{\overrightarrow{\boldsymbol{b}}}\subseteq D_{\overrightarrow{\boldsymbol{b}}}$, we have

$$d(|\mu_1|+\cdots+|\mu_{j-1}|+1)<\cdots<d(d^{-1}(n+1))=n+1<\cdots<d(|\mu_1|+\cdots+|\mu_j|).$$

This is a contradiction. Hence we have proved (A),(B),(C),(D) (for unique $j=\xi(d^{-1}(n+1))$). Since now we know the explicit form of d characterized by (A),(B),(C),(D), (6) follows by the routine calculation. $\square$

Now we assume $L(d)>0$ and d be the form in Sublemma 5.4 for uniquely determined j. By restricting to the subgroup $U_{1,d}\subseteq dYd^{-1}\cap Z$, we have

$$
\begin{aligned}
L(d)&\le\dim\mathrm{Hom}_{U_{1,d}}\left(\mathsf{Res}^{dYd^{-1}\cap Z}_{U_{1,d}}{}^{d}(\mathsf{Res}^{Y}_{Y\cap d^{-1}Zd}\widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau)\otimes\widetilde{D}(\overrightarrow{\boldsymbol{\mu}})),\mathsf{Res}^{Z}_{U_{1,d}}{}^{t^{-1}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau)\otimes\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}}))\right)\\
&=\dim\mathrm{Hom}_{U_{1,d}}\left({}^{\delta}(\mathsf{Res}^{\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}}_{U_{2,d}}(D(\overrightarrow{\boldsymbol{\mu}})^{\oplus\dim\widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau)})),\mathsf{Res}^{\mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}}_{U_{1,d}}({}^{T^{-1}}D(\overrightarrow{\boldsymbol{\lambda}})^{\oplus\dim\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau)})\right)\\
&=\dim\mathrm{Hom}_{U_{1,d}}\left({}^{\delta}(\mathsf{Res}^{\mathfrak{S}_{\overrightarrow{\boldsymbol{b}}}}_{U_{2,d}}D(\overrightarrow{\boldsymbol{\mu}})),\mathsf{Res}^{\mathfrak{S}_{\overrightarrow{\boldsymbol{c}}}}_{U_{1,d}}({}^{T^{-1}}D(\overrightarrow{\boldsymbol{\lambda}}))\right)
\end{aligned}
$$

where $T \stackrel{\text{def}}{=} t|_{\mathfrak{S}_{\vec{\boldsymbol{a}}}} : \mathfrak{S}_{\vec{\boldsymbol{a}}} \xrightarrow{\sim} \mathfrak{S}_{\vec{\boldsymbol{c}}}$ and $\delta \stackrel{\text{def}}{=} \varrho_d|_{U_{1,d}} : U_{1,d} \xrightarrow{\sim} U_{2.d}$. Note that here we use the fact that $\mathsf{Res}^{Y}_{U_{2,d}}(\widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau))$ is a trivial $U_{2,d}$-module and $\mathsf{Res}^{Z}_{U_{1,d}}({}^{t^{-1}}\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau))$ is a trivial $U_{1,d}$-module because (any irreducible FG-module is 1-dimensional and hence) $\dim \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau) = \dim \widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau) = 1$ (see Definition 4.5).

By the explicit form of d characterized by (A),(B),(C),(D) in Sublemma 5.4, and (6), we have an isomorpshim as F-vector spaces

(8)

$$\begin{aligned}&\mathrm{Hom}_{U_{1,d}}\left({}^{\delta}(\mathsf{Res}^{\mathfrak{S}_{\vec{\boldsymbol{b}}}}_{U_{2,d}} D(\overrightarrow{\boldsymbol{\mu}})), \mathsf{Res}^{\mathfrak{S}_{\vec{\boldsymbol{c}}}}_{U_{1,d}}({}^{T^{-1}}D(\overrightarrow{\boldsymbol{\lambda}}))\right)\\ &\cong \mathrm{Hom}_{\mathfrak{S}_{|\lambda_1|}}(D^{\mu_1}_F, D^{\lambda_1}_F)\otimes\cdots\otimes \mathrm{Hom}_{\mathfrak{S}_{|\lambda_j|}}(\mathsf{Res}^{\mathfrak{S}_{|\lambda_j|+1}}_{\mathfrak{S}_{|\lambda_j|}} D^{\mu_j}_F, D^{\lambda_j}_F)\otimes\cdots\otimes \mathrm{Hom}_{\mathfrak{S}_{|\lambda_\alpha|}}(D^{\mu_\alpha}_F, D^{\lambda_\alpha}_F).\end{aligned}$$

Applying (classical or Kleshchev's modular) branching rule for the symmetric groups, we have just proven that

- (2) holds.
- if the equality for (2) holds then (3) holds.

So it remains to show that the equality for (2) holds if (3) holds.

Sublemma 5.5. *Let $\mathcal{G}$ be a group and $\mathcal{G}_1, \mathcal{G}_2$ be its subgroups such that $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \{1_{\mathcal{G}}\}$. Suppose we are given 4 representations of G*

$$\begin{cases}\rho_i : \mathcal{G} \longrightarrow \mathrm{GL}_F(\mathcal{V}_i) & (i=1,2)\\ \psi_i : \mathcal{G} \longrightarrow \mathrm{GL}_F(\mathcal{W}_i) & (i=1,2)\end{cases}$$

such that for any $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$, we have

$$\begin{cases}\rho_1(g_1g_2) = \rho_1(g_1), & \rho_2(g_1g_2) = \rho_2(g_2)\\ \psi_1(g_1g_2) = \psi_1(g_1), & \psi_2(g_1g_2) = \psi_2(g_2).\end{cases}$$

Then we have the inequality

$$\begin{aligned}&\dim \mathrm{Hom}_{\mathcal{G}}(\mathcal{V}_1\otimes\mathcal{V}_2, \mathcal{W}_1\otimes\mathcal{W}_2)\\ &\geq \dim \mathrm{Hom}_{\mathcal{G}_1}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1}\mathcal{V}_1, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1}\mathcal{W}_1)\cdot \dim \mathrm{Hom}_{\mathcal{G}_2}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2}\mathcal{V}_2, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2}\mathcal{W}_2).\end{aligned}$$

Proof. Note that there is a natural injection (between F-vector spaces)

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{G}_1}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1}\mathcal{V}_1, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1}\mathcal{W}_1)\otimes \mathrm{Hom}_{\mathcal{G}_2}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2}\mathcal{V}_2, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2}\mathcal{W}_2)\\ &\hookrightarrow \mathrm{Hom}_{\mathcal{G}}(\mathcal{V}_1\otimes\mathcal{V}_2, \mathcal{W}_1\otimes\mathcal{W}_2)\end{aligned}$$

that sends $\varphi_1\otimes\varphi_2$ to $\varphi_1\otimes\varphi_2$. □

Let us assume that (3) holds. Put $j = \gamma$ and take $d \in D_{\vec{\boldsymbol{c}},\vec{\boldsymbol{b}}}$ characterized by (A),(B),(C),(D) in Sublemma 5.4. As in the above discussion, we have only to show that $L(d) = 1$. Apply Sublemma 5.5 under

$$\begin{cases}\mathcal{G} = dYd^{-1}\cap Z = G^nU_{1,d}(\subseteq X), & \mathcal{G}_1 = G^n(\subseteq \mathcal{G}), \quad \mathcal{G}_2 = U_{1,d}(\subseteq \mathcal{G})\\ \mathcal{V}_1 = {}^{d}(\mathsf{Res}^{Y}_{d^{-1}\mathcal{G}d}\widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau)), & \mathcal{V}_2 = {}^{d}(\mathsf{Res}^{Y}_{d^{-1}\mathcal{G}d}\widetilde{D}(\overrightarrow{\boldsymbol{\mu}}))\\ \mathcal{W}_1 = \mathsf{Res}^{Z}_{\mathcal{G}}({}^{t^{-1}}\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau)), & \mathcal{W}_2 = \mathsf{Res}^{Z}_{\mathcal{G}}({}^{t^{-1}}\widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})))\end{cases}$$

and we have

$$
\begin{aligned}
L(d) &\overset{(5)}{=} \dim \operatorname{Hom}_{dYd^{-1}\cap Z} \left({}^{d}(\mathsf{Res}^{Y}_{Y\cap d^{-1}Zd}\, \widetilde{E}(\overrightarrow{\boldsymbol{b}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\mu}})), \mathsf{Res}^{Z}_{dYd^{-1}\cap Z}\, {}^{t^{-1}}(\widetilde{E}(\overrightarrow{\boldsymbol{a}};\tau) \otimes \widetilde{D}(\overrightarrow{\boldsymbol{\lambda}})) \right) \\
&= \dim \operatorname{Hom}_{\mathcal{G}}(\mathcal{V}_1 \otimes \mathcal{V}_2, \mathcal{W}_1 \otimes \mathcal{W}_2) \\
&\geq \dim \operatorname{Hom}_{\mathcal{G}_1}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1} \mathcal{V}_1, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_1} \mathcal{W}_1) \cdot \dim \operatorname{Hom}_{\mathcal{G}_2}(\mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2} \mathcal{V}_2, \mathsf{Res}^{\mathcal{G}}_{\mathcal{G}_2} \mathcal{W}_2) \overset{(7),(8)}{=} 1.
\end{aligned}
$$

Because the converse inequality (2) has been already established, we reach to the conclusion that $L(d) = 1$. $\square$

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